Ch. 13 Vector Analysis

13.1 Properties of a Vector Field: Divergence and Curl

A vector field assigns a vector to each point in a space, useful in modeling forces, velocities, and flows. The divergence measures a rate of outward flow from a point (useful for density and charge), while the curl assesses the rotation around a point (useful in fluid flow and magnetism). The concepts are widely applied in fields like fluid dynamics to describe and predict motion patterns.

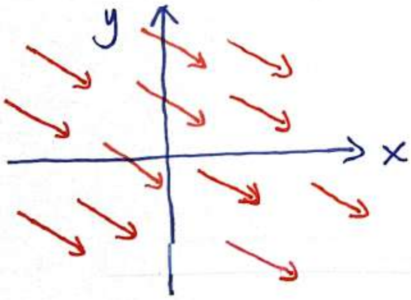
A vector field in 3D is a function F that assigns a vector to each point in its domain. It takes the form

$$\boxed{\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + K(x,y,z)\,\vec{k}},$$

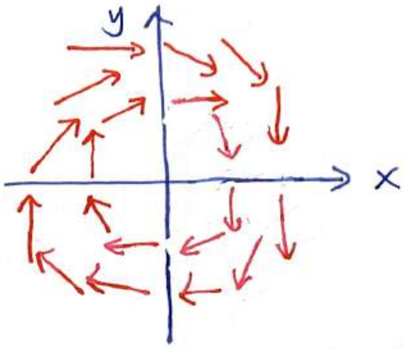
where M, N, and K are the components. A continuous vector field has continuous components, while a differentiable vector field has existing partial derivatives for all components.

Vector fields are usually computer-generated since they are hard to draw by hand. In fluid dynamics, they describe fluid flows: one-dimensional flows vary in one direction, two-dimensional in a plane, and three-dimensional in all directions. A flow is irrotational if its angular velocity is zero and rotational otherwise.

Irrotational flows are sometimes called parallel, and rotational ones circular, though these terms are not precise.



a parallel fluid flow



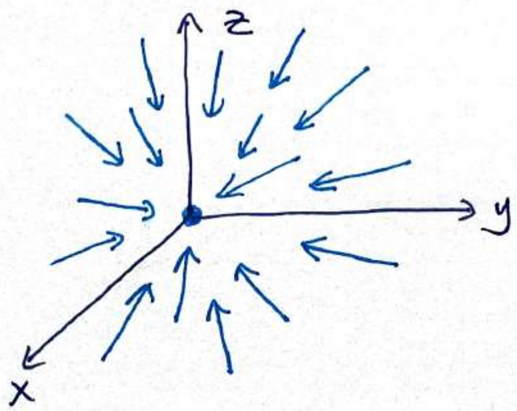
a rotational flow

In fluid dynamics, a flow is steady if its time derivatives are zero; otherwise, it's unsteady. A flow can be both irrotational and unsteady.

Gravitational, electrical, and magnetic vector fields are important in physics. According to Newton's law, a mass m at the origin exerts a gravitational force on a unit mass at $P(x,y,z)$ given by

$$\vec{F}(x,y,z) = -\frac{Gm}{(x^2+y^2+z^2)^{3/2}}(x\vec{i}+y\vec{j}+z\vec{k}),$$

where G is the gravitational constant. This field, directed toward the origin, is known as a central force field.



a center force field

Divergence :

Divergence of a differentiable vector field

$$\vec{F}(x,y,z) = \langle p(x,y,z), q(x,y,z), r(x,y,z) \rangle$$

is defined by

$$\boxed{\operatorname{div} \vec{F} = p_x + q_y + r_z.}$$

$$\left\{ p_x = \frac{\partial p}{\partial x},\ q_y = \frac{\partial q}{\partial y},\ r_z = \frac{\partial r}{\partial z}. \right\}$$

Ex. Find the divergence of the vector field

$$\vec{F}(x,y,z) = \langle \cos x,\ x^4 y^3,\ \ln(x^2 y^2 z^2) \rangle.$$

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(\cos x) + \frac{\partial}{\partial y}(x^4 y^3) + \frac{\partial}{\partial z}\left[\ln(x^2 y^2 z^2)\right]$$

$$= -\sin x + 3x^4 y^2 + \frac{2x^2 y^2 z}{x^2 y^2 z^2}$$

$$\Rightarrow \operatorname{div} \vec{F} = -\sin x + 3x^4 y^2 + \frac{2}{z}.$$

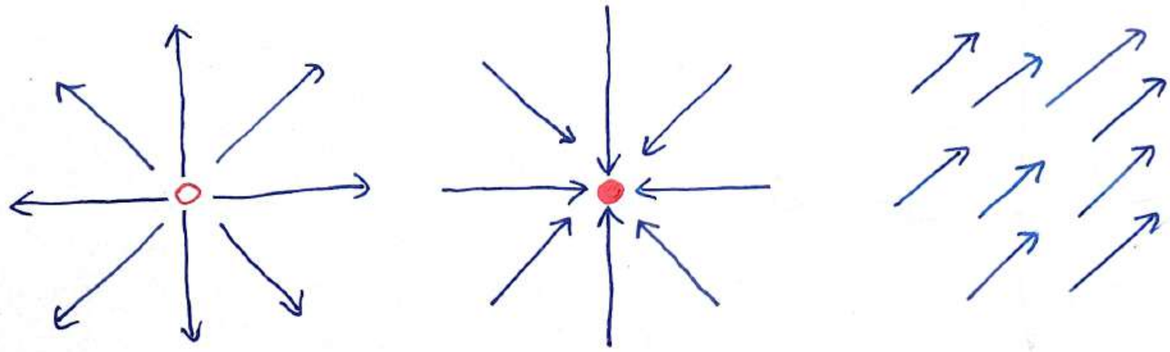
Consider the vector field $\vec{F}(x,y,z) = \langle p, q, r \rangle$ representing the velocity of a fluid with density $\rho(x,y,z)$ in a region R of 3D. The vector field $\rho \vec{F}$, called the flux density and denoted by D, represents the "mass flow" of the liquid.

Assuming no external forces act on the fluid,

$\operatorname{div} D = -\dfrac{\partial \rho}{\partial t}$, which represents the continuity equation.

When $\operatorname{div} D = 0$, the flow is incompressible, meaning the density remains unchanged within a moving volume.

If $D>0$, the point is a source, and if $D<0$, it is a sink. These principles apply to any vector field.



div $D>0$ — fluids flows away from a source point

div $D<0$ — fluid moves toward a sink point

div $D=0$ — fluid remains incompressible

Remember del operator (also known as nabla):

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}.$$

The gradient of a differentiable function $f=f(x,y,z)$ is given by

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}.$$

The divergence of a vector field $\vec{F} = p\vec{i} + q\vec{j} + r\vec{k}$ is

$$\boxed{\nabla \cdot \vec{F} = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z}}$$

Curl:

The curl of a differentiable vector field $\vec{F} = \langle p,q,r\rangle$ is

$$\boxed{\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ p & q & r \end{vmatrix}.}$$

A vector field is irrotational if its curl is zero. In fluid dynamics, the curl of the velocity field is called the vorticity, and if it is zero, the flow is irrotational.

$$\operatorname{div} \vec{F} = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z}$$

$$= \frac{\partial}{\partial x}(p) + \frac{\partial}{\partial y}(q) + \frac{\partial}{\partial z}(r)$$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \bullet \left(p\vec{i} + q\vec{j} + r\vec{k}\right)$$

$$\Rightarrow \boxed{\operatorname{div} \vec{F} = \nabla \bullet \vec{F}}$$

$$\left\{\begin{aligned} \operatorname{div} \vec{F} &= \nabla \bullet \vec{F} \\ \operatorname{curl} \vec{F} &= \nabla \times \vec{F} \end{aligned}\right\}$$

Ex. Find the curl of the vector field

$$\vec{G} = x\cos y\,\vec{i} + y\sin z\,\vec{j} + z\cos x\,\vec{k}.$$

$$\operatorname{curl} \vec{G} = \nabla \times \vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x\cos y & y\sin z & z\cos x \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(z\cos x) - \frac{\partial}{\partial z}(y\sin z)\right]\vec{i}$$

$$- \left[\frac{\partial}{\partial x}(z\cos x) - \frac{\partial}{\partial z}(x\cos y)\right]\vec{j}$$

$$+ \left[\frac{\partial}{\partial x}(y\sin z) + \frac{\partial}{\partial y}(x\cos y)\right]\vec{k}$$

$$= \left[0 - y\cos z\right]\vec{i} - \left[-z\sin x - 0\right]\vec{j} + \left[0 + x(-\sin y)\right]\vec{k}$$

$$= -y\cos z\,\vec{i} + z\sin x\,\vec{j} - x\sin y\,\vec{k}.$$

Ex. For $\vec{V} = 3\vec{i} - 4\vec{j} + 5\vec{k}$, find the following

a- div $\vec{V}$,

b- curl $\vec{V}$.

a- $\operatorname{div} \vec{V} = \frac{\partial}{\partial x}(3) + \frac{\partial}{\partial y}(-4) + \frac{\partial}{\partial z}(5)$

$= 0$

b- $$\operatorname{curl} \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3 & -4 & 5 \end{vmatrix}$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k} = \vec{0} = \langle 0, 0, 0 \rangle$$

Laplacian Operator

The Laplacian of a function $h(x,y,z)$ with continuous first and second partial derivatives is given by

$$\nabla^2 h = \nabla \cdot \nabla h$$

$$= \operatorname{div} \nabla h$$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(\frac{\partial h}{\partial x}\vec{i} + \frac{\partial h}{\partial y}\vec{j} + \frac{\partial h}{\partial z}\vec{k}\right)$$

$$= \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}$$

$$\Rightarrow \nabla^2 h = h_{xx} + h_{yy} + h_{zz}$$

$\left\{ \nabla^2 h = 0 \text{ Laplace's equation} \right\}$

and any function that satisfies this equation within a region D is termed harmonic in D.

Ex. Determine $h(x,y) = e^y \cos x$ is harmonic or not.

$$h_x = -e^y \sin x \quad \Big| \quad h_y = e^y \cos x$$
$$h_{xx} = -e^y \cos x \quad \Big| \quad h_{yy} = e^y \cos x$$

$\Rightarrow h_{xx} + h_{yy} = (-e^y \cos x) + (e^y \cos x) = 0$

$\Rightarrow h(x,y)$ is a harmonic function and $\nabla^2 h = 0$.

Ex. Let $\vec{H} = \langle yz, 3xz, 2xy \rangle$. Find the following

a- $\operatorname{div} \vec{H}$,

b- $\operatorname{curl} \vec{H}$,

c- $\operatorname{div} \operatorname{curl} \vec{H}$.

a- $\operatorname{div} \vec{H} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(3xz) + \frac{\partial}{\partial z}(2xy)$

$= 0 + 0 + 0 = 0.$

b- $\operatorname{curl} \vec{H} = \nabla \times \vec{H} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 3xz & 2xy \end{vmatrix}$

$= \left[\frac{\partial}{\partial y}(2xy) - \frac{\partial}{\partial z}(3xz)\right]\vec{i}$

$- \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial z}(yz)\right]\vec{j} + \left[\frac{\partial}{\partial x}(3xz) - \frac{\partial}{\partial y}(yz)\right]\vec{k}$

$= [2x - 3x]\vec{i} - [2y - y]\vec{j} + [3z - z]\vec{k}$

$= -x\,\vec{i} - y\,\vec{j} + 2z\,\vec{k} = \langle -x, -y, 2z \rangle.$

c- $\operatorname{div} \operatorname{curl} \vec{H} = \operatorname{div}(\langle -x, -y, 2z \rangle)$

$= \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(2z)$

$= -1 + (-1) + 2 = 0.$

13.2 Line Integrals

The line integral of $f(x,y,z)$ over a smooth curve C with parametric equations

$$x = x(t)\ ,\quad y = y(t)\ ,\quad z = z(t)$$

is given by

$$\int_C f(x,y,z)\,ds = \lim_{\|\Delta s\| \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta s_k \ .$$

For closed curves, the line integral is often written as

$$\boxed{\oint_C f(x,y,z)\,ds \ .}$$

If f is continuous on C, the line integral exists.

Since C is smooth, we have

$$ds = \sqrt{(x')^2 + (y')^2 + (z')^2}\ dt \ .$$

Thus, the line integral is

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt \ .}$$

The line integral over a piecewise smooth curve C, composed of subarcs $C_1, C_2, \ldots, C_n$, is the sum of integrals over each subarc :

$$\int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds + \ldots + \int_{C_n} f(x,y,z)\,ds \ .$$

Ex. Let C be the curve which is the union of two line segments: the first segment goes from $(1,1)$ to $(5,3)$, and the second segment goes from $(5,3)$ to $(9,1)$. Compute the line integral

$$\int_C 3dy - dx .$$

C_1: from $(1,1)$ to $(5,3)$.

Parametrization of segment C_1 (with $0 \leq t \leq 1$) is

$$x_1(t) = 1 + 4t , \quad y_1(t) = 1 + 2t ,$$

$$\Rightarrow \quad \frac{dx_1}{dt} = 4 , \quad \frac{dy_1}{dt} = 2 .$$

$$\Rightarrow \int_{C_1} 3dy - dx = \int_0^1 \left(3\frac{dy_1}{dt} - \frac{dx_1}{dt}\right) dt = \int_0^1 \left(3(2) - 4\right) dt$$

$$= \int_0^1 2\, dt = 2 .$$

C_2: from $(5,3)$ to $(9,1)$.

$$\Rightarrow \quad x_2(t) = 5 + 4t , \quad y_2(t) = 3 - 2t \quad (0 \leq t \leq 1),$$

$$\Rightarrow \quad \frac{dx_2}{dt} = 4 , \quad \frac{dy_2}{dt} = -2 .$$

$$\Rightarrow \int_{C_2} 3dy - dx = \int_0^1 \left(3\frac{dy_2}{dt} - \frac{dx_2}{dt}\right) dt = \int_0^1 \left(3(-2) - 4\right) dt$$

$$= \int_0^1 (-10)\, dt = -10 .$$

$$\Rightarrow \int_C 3dy - dx = \int_{C_1} 3dy - dx + \int_{C_2} 3dy - dx$$

$$= 2 + (-10) = -8 .$$

Ex. Evaluate the line integral over the curve C, where $x=\cos t$, $y=\sin t$, for $0 \le t \le \frac{\pi}{2}$:

$$\int_C (4x-3y)\,ds .$$

$\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$.

$$\Rightarrow ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{\underbrace{(-\sin t)^2 + (\cos t)^2}_{=1}}\,dt = dt$$

$$\int_C (4x-3y)\,ds = \int_0^{\frac{\pi}{2}} (4\cos t - 3\sin t)\,dt$$

$$= \left[4\sin t - (-3\cos t)\right]\Big|_0^{\pi/2}$$

$$= \left(4\sin\left(\tfrac{\pi}{2}\right) + 3\cos\left(\tfrac{\pi}{2}\right)\right) - \left(4\sin(0) + 3\cos(0)\right)$$

$$= \left(4\cdot(1) + 3\cdot(0)\right) - \left(4\cdot(0) + 3\cdot(1)\right)$$

$$= 4 - 3 = 1 .$$

Properties of Line Integrals:

For a scalar function f defined on a piecewise smooth, orientable curve C :

- $\boxed{\int_C \alpha \cdot f\,ds = \alpha \int_C f\,ds}$, $(\alpha \in \mathbb{R})$ (constant multiple)
- $\boxed{\int_C (f_1 + f_2)\,ds = \int_C f_1\,ds + \int_C f_2\,ds}$ (sum) (f_1, f_2 are scalar functions on C)
- $\boxed{\int_{-C} f\,ds = -\int_C f\,ds}$ ($-C$ is C transversed in the reverse direction (opposite direction))

- $$\boxed{\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \dots + \int_{C_n} f\,ds}$$ (subdivision)

($C = C_1 \cup C_2 \cup \dots \cup C_n$, (C_i : orientable subarcs) where the subarcs share only their endpoints.)

Line integrals with respect to x, y, and z are given by

$$\int_C f(x,y,z)\,dx = \int_a^b f(x(t),y(t),z(t))\,x'(t)\,dt,$$

$$\int_C g(x,y,z)\,dy = \int_a^b g(x(t),y(t),z(t))\,y'(t)\,dt,$$

$$\int_C h(x,y,z)\,dz = \int_a^b h(x(t),y(t),z(t))\,z'(t)\,dt.$$

Then, we get a line integral :

$$\int_C \left[f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz\right].$$

Ex. Evaluate the line integral $\int_C (-x\,dx + y\,dy - z\,dz)$, where the curve C is given by the parametric equations

$$x = t, \quad y = t^2, \quad z = t^3, \qquad 0 \le t \le 1.$$

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2t, \quad \frac{dz}{dt} = 3t^2$$

$$\Rightarrow \int_C (-x\,dx + y\,dy - z\,dz = \int_0^1 (-t\,dt + t^2\,2t\,dt - t^3\,3t^2\,dt)$$

$$= \int_0^1 (-t\,dt + 2t^3\,dt - 3t^5\,dt)$$

$$= \left(-\frac{t^2}{2} + 2\frac{t^4}{4} - 3\frac{t^6}{6}\right)\Big|_0^1$$

$$= -\frac{1}{2} + \frac{1}{2} - \frac{1}{2} = -\frac{1}{2}$$

The line integral of a vector field

$$\vec{F}(x,y,z) = u(x,y,z)\,\vec{i} + v(x,y,z)\,\vec{j} + w(x,y,z)\,\vec{k}$$

along a piecewise smooth orientable curve C with parametric representation

$$\vec{R}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$

for $a \le t \le b$ is defined as

$$\int_C \vec{F}\cdot d\vec{R} = \int_a^b \Big[u(x(t),y(t),z(t))\frac{dx}{dt} + v(x(t),y(t),z(t))\frac{dy}{dt} + w(x(t),y(t),z(t)\frac{dz}{dt} \Big] dt .$$

This can be written as

$$\int_C \vec{F}\cdot d\vec{R} = \int_C (u\,dx + v\,dy + w\,dz)$$
$$= \int_C \vec{F}(\vec{R}(t))\cdot \vec{R}'(t)\,dt .$$

We know $dR = R'(t)\,dt$, then

$$\int_C \vec{F}\cdot dR = \int_a^b \vec{F}(R(t))\cdot R'(t)\,dt$$
$$= \int_a^b \vec{F}(R(t))\cdot \frac{R'(t)}{\|R'(t)\|}\,\|R'(t)\|\,dt$$

$$\Rightarrow \int_C \vec{F}\cdot dR = \int_a^b \vec{F}(R(t))\cdot \vec{T}(t)\,\|R'(t)\|\,dt .$$

Ex. Evaluate the line integral over the curve C:

$$x = e^{-t}\cos(2t),\; y = e^{-t}\sin(2t),\; t \in [0, \tfrac{\pi}{4}],$$

$$\int_C (x^2+y^2)\,ds.$$

$$x^2+y^2 = [e^{-t}\cos(2t)]^2 + [e^{-t}\sin(2t)]^2$$
$$= e^{-2t}\cos^2(2t) + e^{-2t}\sin^2(2t)$$
$$= e^{-2t}[\underbrace{\cos^2(2t)+\sin^2(2t)}_{=1}]$$

$$\Rightarrow x^2+y^2 = e^{-2t}$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\frac{dx}{dt} = -e^{-t}\cos(2t) - 2e^{-t}\sin(2t)$$
$$\frac{dx}{dt} = e^{-t}(-\cos(2t) - 2\sin(2t))$$
$$\frac{dy}{dt} = -e^{-t}\sin(2t) + 2e^{-t}\cos(2t)$$
$$\frac{dy}{dt} = e^{-t}(-\sin(2t) + 2\cos(2t))$$

$$\left(\frac{dx}{dt}\right)^2 = e^{-2t}(-\cos(2t) - 2\sin(2t))^2$$
$$\left(\frac{dx}{dt}\right)^2 = e^{-2t}(\cos^2(2t) + 4\sin(2t)\cos(2t) + 4\sin^2(2t))$$
$$\left(\frac{dy}{dt}\right)^2 = e^{-2t}(-\sin(2t) + 2\cos(2t))^2$$
$$\left(\frac{dy}{dt}\right)^2 = e^{-2t}(\sin^2(2t) - 4\sin(2t)\cos(2t) + 4\cos^2(2t))$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{e^{-2t}(1+4)}\,dt = \sqrt{5}\,e^{-t}\,dt$$

$$\Rightarrow \int_C (x^2+y^2)\,ds = \int_0^{\pi/4} e^{-2t}.\sqrt{5}\,e^{-t}\,dt = \sqrt{5}\int_0^{\pi/4} e^{-3t}\,dt$$
$$= -\frac{\sqrt{5}}{3}e^{-3t}\Big|_0^{\pi/4} = \frac{\sqrt{5}}{3}\left(1 - e^{-3\pi/4}\right).$$

Ex. A wire has the shape of the curve

$$x = \sin t, \quad y = \cos t, \quad z = t, \qquad t \in [0, \pi].$$

The wire has a density function $\rho(x,y,z) = x^2 + y^2 + z^2$ at each point (x,y,z). What is the mass of the wire?

To find the mass, we need to calculate the line integral $m = \int_C \rho(x,y,z)\,ds = \int_C \rho \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$.

$$\Rightarrow \frac{dx}{dt} = \cos t, \quad \frac{dy}{dt} = -\sin t, \quad \frac{dz}{dt} = 1.$$

$$\Rightarrow m = \int_C \rho\,ds = \int_0^{\pi} (\sin^2 t + \cos^2 t + t^2)\sqrt{(\cos t)^2 + (-\sin t)^2 + 1}\,dt$$

$$= \int_0^{\pi} (1 + t^2)\sqrt{2}\,dt$$

$$m = \sqrt{2}\left(t + \frac{t^3}{3}\right)\Big|_0^{\pi} = \sqrt{2}\left(\pi + \frac{\pi^3}{3}\right).$$

D: displacement

Line integration is crucial in computing work, $(W = \vec{F} \cdot D)$ especially when the force field is variable. For an object moving along a curve C, parametrized by $\vec{R}(t)$, the work done is sum of force contributions along small segments the curve. By partitioning the curve into subdivisions, the total work is calculated as the line integral of the force field along C.

The work **W** done by a continuous force field $\vec{F}$ along a smooth curve C is defined by the line integral

$$\boxed{W = \int_C \vec{F} \cdot \vec{T}\,ds}, \quad \left\{ \vec{T} = \frac{d\vec{R}}{ds}, \ \vec{R}: \text{position vector} \right\}$$

where T determines the unit tangent vector on C.

$$W = \int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot \frac{d\vec{R}}{ds}\, ds = \int_C \vec{F} \cdot d\vec{R}$$

W: work

$\vec{T}$: unit tangent vector

$\vec{R}$: position vector.

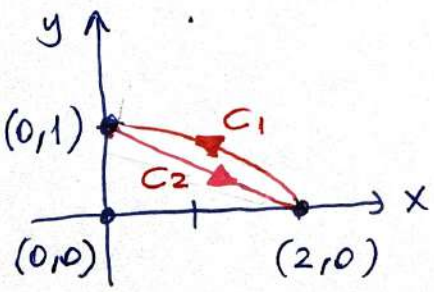
Ex. (From the textbook)

Calculate the work done by the force field

$$\vec{F} = y^2\,\vec{i} + 2(x+1)\,\vec{j}$$

as an object moves from $(2,0)$ to $(0,1)$ along the ellipse $x^2+4y^2=4$, and then back to $(2,0)$ along the line segment.

If C represents the path of the moving object, the work done is $\boxed{W = \oint_C \vec{F} \cdot d\vec{R}} = \int_{C_1} \vec{F} \cdot d\vec{R}_1 + \int_{C_2} \vec{F} \cdot d\vec{R}_2$.



C_1 : elliptical curve

$x = 2\cos t$, $y = \sin t$, $t \in [0, \frac{\pi}{2}]$.

$\vec{R}_1 = \langle 2\cos t, \sin t \rangle$

$\vec{R}_1{}' = \langle -2\sin t, \cos t \rangle$

$$W_1 = \int_{C_1} \vec{F} \cdot d\vec{R}_1 = \int_0^{\pi/2} \vec{F}(\vec{R}_1(t)) \cdot R_1'(t)\, dt$$

$$\Rightarrow W_1 = \int_0^{\pi/2} \left[(\sin t)^2 . (-2\sin t) + 2(2\cos t + 1)\sin t(\cos t)\right] dt$$

$$= \int_0^{\pi/2} \left(-2\sin^3 t + 4\cos^2 t \sin t + 2\sin t \cos t\right) dt$$

$$= \int_0^{\pi/2} \left(-2\sin^2 t + 4\cos^2 t + 2\cos t\right) \sin t\, dt$$

$$= \int_0^{\pi/2} \left(-2(1-\cos^2 t) + 4\cos^2 t + 2\cos t\right) \sin t\, dt$$

$$= \int_0^{\pi/2} \left(6\cos^2 t + 2\cos t - 2\right) \sin t\, dt$$

$$\Rightarrow W_1 = -\int_1^0 (6u^2+2u-2)\,du \qquad \left\{ \begin{array}{l} u=\cos t \\ du=-\sin t\,dt \\ t=0 \Rightarrow u=1 \\ t=\frac{\pi}{2} \Rightarrow u=0 \end{array} \right\}$$

$$= \int_0^1 (6u^2+2u-2)\,du$$

$$= \left(6\frac{u^3}{3} + u^2 - 2u\right)\Big|_0^1$$

$$= 2(1)^3 + (1)^2 - 2(1) = 1.$$

C_2: line segment, $x = 2t$, $y = 1-t$, $t \in [0,1]$.

$$\vec{R_2} = \langle 2t, 1-t \rangle$$

$$\vec{R_2}' = \langle 2, -1 \rangle$$

$$\Rightarrow W_2 = \int_{C_2} \vec{F} \cdot d\vec{R_1} = \int_0^1 \vec{F}(\vec{R_2}(t)) \cdot \vec{R_2}'(t)\,dt$$

$$= \int_0^1 \left[(1-t)^2(2) + 2(2t+1)(1-t)(-1) \right] dt$$

$$= \int_0^1 \left[(1-2t+t^2)2 + 2(2t-2t^2+1-t)(-1) \right] dt$$

$$= \int_0^1 \left[(2-4t+2t^2) + (-4t+4t^2-2+2t) \right] dt$$

$$= \int_0^1 (6t^2-6t)\,dt$$

$$= 6\int_0^1 (t^2-t)\,dt$$

$$= 6\left(\frac{t^3}{3} - \frac{t^2}{2}\right)\Big|_0^1 = 6\left(\frac{1}{3} - \frac{1}{2}\right) = -1$$

$$\Rightarrow W = \oint_C \vec{F} \cdot d\vec{R} = \underbrace{\int_{C_1} \vec{F} \cdot d\vec{R_1}}_{=1} + \underbrace{\int_{C_2} \vec{F} \cdot d\vec{R_2}}_{=(-1)}$$

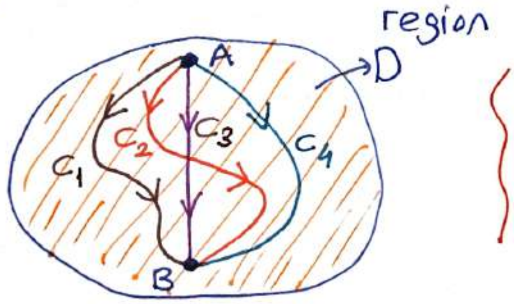
$$= 1 + (-1)$$

$$\Rightarrow \boxed{W = 0}.$$

{The work done by a force along any closed path is 0, indicating that the work is conservative.}

13.3 The Fundamental Theorem and Path Independence

A line integral $\int_C \vec{F} \cdot d\vec{R}$ is independent of path in a region if its value is the same for all paths between two points.



{ A, B : endpoints of the curves }

The fundamental theorem of calculus states that for a continuous function f, $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is an antiderivative of f.

For functions of two or three variables, the analogue is the gradient.

The fundamental theorem for the line integrals states that if $\vec{F} = \nabla f$, then

$$\boxed{\int_C \vec{F} \cdot d\vec{R} = f(B) - f(A)}, \quad \left\{ \begin{array}{l} B = R(b) \\ A = R(a) \end{array} \right\}$$

where A and B are the endpoints of the curve C.

$$\int_C \vec{F} \cdot d\vec{R} = \int_C \nabla f \cdot d\vec{R}$$

Ex. Compute the line integral $\int_C \vec{F} \cdot d\vec{R}$, where $\vec{F} = \nabla(e^x \cos y - xy^2 - 3y)$ and C is the path given by $\vec{R}(t) = \langle t^2 \cos(\pi t), -2\sin(\pi t) \rangle$ for $t \in [0,1]$.

$f(x,y) = e^x \cos y - xy^2 - 3y$,

$f_x = \frac{\partial f}{\partial x} = e^x \cos y - y^2$

$f_y = \frac{\partial f}{\partial y} = -e^x \sin y - 2xy - 3$

$\Rightarrow \nabla f = \langle f_x, f_y \rangle$

$= \langle e^x \cos y - y^2, -e^x \sin y - 2xy - 3 \rangle$

$\vec{F} = \nabla f = \langle e^x \cos y - y^2, -e^x \sin y - 2xy - 3 \rangle$

The path C is given by $\vec{R}(t) = \langle t^2 \cos(\pi t), -2\sin(\pi t) \rangle$.

Then, at $t=0$, $\vec{R}(0) = \langle 0,0 \rangle$,

at $t=1$, $\vec{R}(1) = \langle -1, 0 \rangle$.

At $A = (0,0)$: $f(0,0) = e^0 \cos(0) - 0.0^2 - 3.0 = 1$.

At $B = (-1,0)$: $f(-1,0) = e^{-1} \cos(0) - (-1).0^2 - 3.0 = \frac{1}{e}$.

Since $\vec{F} = \nabla f$, we can use the fundamental theorem of line integrals, which states that

$$\int_C \vec{F} \cdot d\vec{R} = f(B) - f(A)$$

$$= f(-1,0) - f(0,0)$$

$$= \frac{1}{e} - 1.$$

Conservative Vector Fields

A vector field $\vec{F}$ is considered conservative in a region D if there exists a scalar function f such that

$$\boxed{\vec{F} = \nabla f}$$

in D. The function f is referred to as the scalar potential of $\vec{F}$ in D.

The term "conservative" originates from physics and is associated with the law of conservation of energy.

Ex. Verify that the vector field

$$\vec{F}(x,y) = y^2\vec{i} + 2xy\vec{j}$$

is conservative and find its scalar potential $f(x,y)$.

The vector field $\vec{F}$ can be written as

$$\vec{F}(x,y) = u(x,y)\vec{i} + v(x,y)\vec{j}.$$

$$\Rightarrow \begin{cases} u(x,y) = y^2, \\ v(x,y) = 2xy. \end{cases}$$

$$\left.\begin{aligned} u_y &= \frac{\partial u(x,y)}{\partial y} = 2y \\ v_x &= \frac{\partial v(x,y)}{\partial x} = 2y \end{aligned}\right] \Rightarrow u_y = v_x = 2y$$

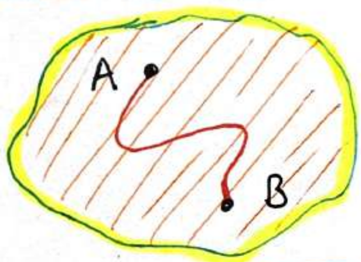
$$\nabla f = \langle f_x, f_y \rangle = \langle y^2, 2xy \rangle = \vec{F}$$

$\Rightarrow$ F is conservative.

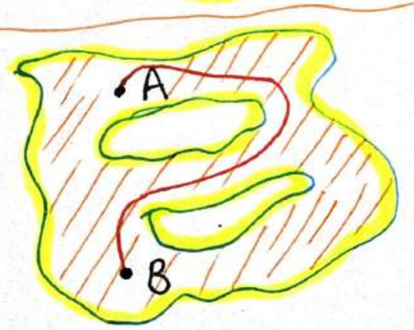
To determine if a vector field $\vec{F}$ is conservative, we use a simple criterion for a region D in the plane with two properties:

1) Any two points A and B in D can be connected by a piecewise-smooth curve within D.
2) Every closed curve in D surrounds only points that are also within D.

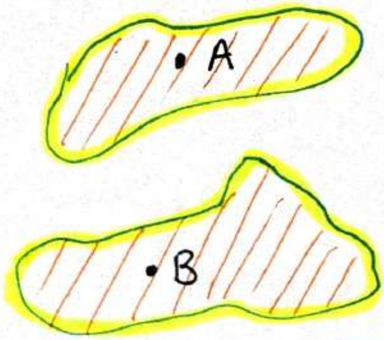
A region meeting (1) is referred to as connected, and if it also adheres to condition (2), it is known as simply connected, implying it has no holes.



A region without any holes, known as simply connected.



A connected region that contains one or more holes, making it not simply connected.



A region that is disconnected, consisting of seperate parts.

A vector field $\vec{F}(x,y) = u(x,y)\vec{i} + v(x,y)\vec{j}$ is **conservative** in an open, simply connected region D if and only if $\boxed{\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}}$ holds everywhere in D.

Ex. Verify that the vector field

$$\vec{F}(x,y) = \left(2xe^{xy} + x^2ye^{xy}\right)\vec{i} + \left(x^3e^{xy} + 2y\right)\vec{j}$$

is conservative, and find a scalar potantiel function $f(x,y)$ for $\vec{F}(x,y)$.

The vector field $\vec{F}$ is given by

$$\vec{F}(x,y) = u(x,y)\,\vec{i} + v(x,y)\,\vec{j}$$

where

$$u(x,y) = 2xe^{xy} + x^2y\,e^{xy}$$

$$\Rightarrow \underline{u_y} = \left(2x^2e^{xy}\right) + \left(x^2e^{xy} + x^3ye^{xy}\right) = \underline{3x^2e^{xy} + x^3ye^{xy}}$$

$$v(x,y) = x^3e^{xy} + 2y$$

$$\Rightarrow \underline{v_x} = \left(3x^2e^{xy} + x^3ye^{xy}\right) + 0 = \underline{3x^2e^{xy} + x^3ye^{xy}}$$

$$\Rightarrow u_y = v_x \Leftrightarrow F \text{ is conservative} \Rightarrow \vec{F} = \nabla f$$

$$\Rightarrow \vec{F}(x,y) = \nabla f(x,y) = \langle f_x, f_y \rangle$$

$$f_x = 2xe^{xy} + x^2ye^{xy} \Rightarrow f(x,y) = \int\left(2xe^{xy} + x^2ye^{xy}\right)dx$$

or

$$f_y = x^3e^{xy} + 2y \Rightarrow f(x,y) = \int\left(x^3e^{xy} + 2y\right)dy.$$

$$f(x,y) = \int\left(x^3e^{xy} + 2y\right)dy = \frac{x^3}{x}e^{xy} + y^2 + h(x) = \underline{x^2e^{xy} + y^2 + h(x)}$$

$$\Rightarrow f_x = 2xe^{xy} + x^2ye^{xy} + h'(x) = 2xe^{xy} + x^2ye^{xy}$$

{taking C=0}

$$\Rightarrow h'(x) = 0 \Rightarrow \underline{h(x) = c} \Rightarrow \underline{f(x,y) = x^2e^{xy} + y^2 + 0}$$

If $U_y \neq V_x$, then $\vec{F}$ is not conservative.

Theorem 1. In a simply connected region D of $\mathbb{R}^3$, if the vector field $\vec{F}$ and curl $\vec{F}$ are continuous, then $\vec{F}$ is conservative if and only if curl $\vec{F} = 0$.

A vector field $\vec{F}(x,y) = \langle U(x,y), V(x,y) \rangle$ in $\mathbb{R}^2$ can be extended to $\mathbb{R}^3$ as $\langle U(x,y), V(x,y), 0 \rangle = \vec{G}$

It's curl is:

$$\text{Curl}\,\vec{G} = \nabla \times \vec{G} = \langle 0, 0, V_x - U_y \rangle.$$

Then, Theorem 1 becomes the cross-partials test:

$$V_x = U_y \iff \text{curl}\,\vec{G} = 0$$

which confirms $\vec{F}$ is conservative.

Ex. Verify whether the vector field

$$\vec{F}(x,y,z) = \langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 \rangle$$

is conservative in $\mathbb{R}^3$. If it is, find a scalar potential function $\phi(x,y,z)$ such that $\nabla\phi = \vec{F}$.

For $\vec{F} = \langle P, Q, R \rangle$, where

$$P = P(x,y,z) = 2xy^3z^4,$$
$$Q = Q(x,y,z) = 3x^2y^2z^4,$$
$$R = R(x,y,z) = 4x^2y^3z^3,$$

the curl of $\vec{F}$ is

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} \qquad \left\{ \partial_x = \frac{\partial}{\partial x} \right\}$$

Compute each component :

$\vec{i}$ - component : $R_y - Q_z = 12x^2y^2z^3 - 12x^2y^2z^3 = 0$,

$\vec{j}$ - component : $P_z - R_x = 8xy^3z^3 - 8xy^3z^3 = 0$,

$\vec{k}$ - component : $Q_x - P_y = 6xy^2z^4 - 6xy^2z^4 = 0$.

Thus, $\nabla \times \vec{F} = \text{curl}\,\vec{F} = 0 \Rightarrow \vec{F}$ is conservative.

To find ϕ, solve $\nabla\phi = \vec{F}$.

$$\Rightarrow \nabla\phi = \langle \phi_x, \phi_y, \phi_z \rangle = \langle f_1, f_2, f_3 \rangle$$

$$\Rightarrow \begin{cases} \phi_x = f_1 \\ \phi_y = f_2 \\ \phi_z = f_3 \end{cases} \Rightarrow \begin{cases} \phi_x = 2xy^3z^4 \\ \phi_y = 3x^2y^2z^4 \\ \phi_z = 4x^2y^3z^3 \end{cases}$$

$$\Rightarrow \phi(x,y,z) = \int (2xy^3z^4)\,dx = x^2y^3z^4 + h(y,z)$$

$$\Rightarrow \phi_y = \frac{\partial}{\partial y}\left(x^2y^3z^4 + h(y,z)\right) = 3x^2y^2z^4 + \frac{\partial h}{\partial y}$$

$$\Rightarrow \phi_y = 3x^2y^2z^4 = 3x^2y^2z^4 + \frac{\partial h}{\partial y}$$

$$\Rightarrow \frac{\partial h}{\partial y} = 0 \Rightarrow h(y,z) = g(z).$$

$$\Rightarrow \frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z}\left(x^2y^3z^4 + g(z)\right) = 4x^2y^3z^3 + \frac{\partial g}{\partial z}$$

$$\Rightarrow \frac{\partial \phi}{\partial z} = 4x^2y^3z^3 = 4x^2y^3z^3 + \frac{\partial g}{\partial z} \Rightarrow \frac{\partial g}{\partial z} = 0$$

$$\Rightarrow g(z) = c \quad (c = \text{constant}) \Rightarrow \phi(x,y,z) = x^2y^3z^4 + c$$

Ex. Find a potential function for the vector field
$\vec{F}(x,y,z) = \langle 2x\cos y - 2z^3 ,\ 3 + 2ye^z - x^2\sin y ,\ y^2e^z - 6xz^2 \rangle$.

Ex. Evaluate $\int_C \vec{F} \cdot d\vec{R}$, where

$$\vec{F} = \langle 2x^3y^4 + x ,\ 2x^4y^3 + y \rangle ,$$

and curve C is given by $\vec{R}(t) = \langle t\cos(\pi t),\ \sin\left(\frac{\pi t}{2}\right) \rangle$,
$t \in [0,1]$, $t \in \mathbb{R}$.

Independence of Path

The line integral $\int_C \vec{F} \cdot d\vec{R}$ is path-independent in a region D if, for any two points A and B in D, the integral along any smooth curve between them gives the same result.

For a continuous vector field $\vec{F}$ on an open, connected set D, the following are equivalent:

a- F is conservative: $\vec{F} = \nabla f$ for some f.
b- $\int_C \vec{F} \cdot d\vec{R} = 0$ for all closed curves in D.
c- $\int_C \vec{F} \cdot d\vec{R}$ depends only on the endpoints of C.

---

b- $\boxed{\oint_C \vec{F} \cdot d\vec{R} = 0.}$

A=B

C

region D

Let F be conservative in D, and C be any closed curve whose interior is in D. Then, (b) occurs.
Cut C at $A=B$, then
$\oint_C \vec{F} \cdot d\vec{R} = \int_C \vec{F} \cdot d\vec{R} = f(B) - f(A) = 0$.

Remember 13.3 (in 2D):

If the vector field $\vec{F} = \langle F_1, F_2 \rangle$ is conservative on D

$$F_{1y} = F_{2x} \cdot \left\{ \begin{array}{c} \text{or } \vec{F} = \langle u, v \rangle \\ u_y = v_x \end{array} \right\}$$

If conservative find $\vec{F}$ by solving

$$\vec{F} = \int F_1 \, dx + g(y),$$

$$\vec{F} = \int F_2 \, dy + h(x).$$

In 3D: $\vec{F}(x,y,z) = \langle F_1, F_2, F_3 \rangle$

$$F_1 = F_x \begin{array}{l} \xrightarrow{\partial y} F_{1y} = F_{xy} \\ \xrightarrow{\partial z} F_{1z} = F_{xz} \end{array}$$

$$F_2 = F_y \begin{array}{l} \xrightarrow{\partial x} F_{2x} = F_{yx} \\ \xrightarrow{\partial z} F_{2z} = F_{zx} \end{array}$$

$$F_3 = F_z \begin{array}{l} \xrightarrow{\partial x} F_{3x} = F_{zx} \\ \xrightarrow{\partial y} F_{3y} = F_{zy} \end{array}$$

$$\boxed{\begin{array}{l} F_{1y} = F_{2x}, \\ F_{1z} = F_{3x}, \\ F_{2z} = F_{3y}. \end{array}}$$

$$\nabla \times \vec{F} = 0 \Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\{ \begin{array}{l} (F_{3y} - F_{2z})\vec{i} \\ -(F_{3x} - F_{1z})\vec{j} \\ +(F_{2x} - F_{1y})\vec{k} \end{array} \right\} = \langle 0, 0, 0 \rangle$$

$$\left. \begin{array}{l} F_x = F_1 \\ F_y = F_2 \\ F_z = F_3 \end{array} \right\} \overset{\text{find}}{\Rightarrow} \begin{array}{l} \vec{F} = \int F_1 \, dx + g(y,z) \\ \vec{F} = \int F_2 \, dy + h(x,z) \\ \vec{F} = \int F_3 \, dz + l(x,y) \end{array}$$

Ex. $\vec{F} = \langle 20x^3z + 2y^2,\ 4xy,\ 5x^4 + 3z^2 \rangle$

$\vec{F}$ is conservative $\iff \nabla \times \vec{F} = \vec{0} = \langle 0,0,0 \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 20x^3z+2y^2 & 4xy & 5x^4+3z^2 \end{vmatrix} = (0-0)\vec{i} - (20x^3 - 20x^3)\vec{j} + (4y-4y)\vec{k}$$

$$= \langle 0,0,0 \rangle$$

Find $\vec{F}$:

$$\vec{F} = \int (20x^3z + 2y^2)\,dx + \dots = 20\frac{x^4}{4}z + 2y^2x + \dots$$

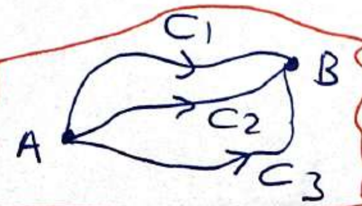
$$\vec{F} = \int 4xy\,dy + \dots = 4x\frac{y^2}{2} + \dots$$

$$\vec{F} = \int (5x^4 + 3z^2)\,dz + \dots = 5x^4z + 3\frac{z^3}{3} + \dots$$

$$\Rightarrow \vec{F} = 5x^4z + 2xy^2 + z^3 + C$$

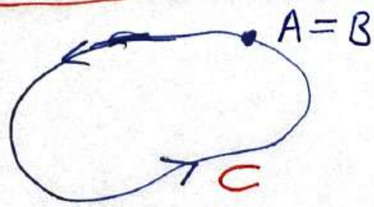
- If $F = \nabla f \Rightarrow$ $\boxed{\int_C F \cdot d\vec{R} = F(B) - F(A)}$

- $\boxed{\int_{C_1} \vec{F} \cdot d\vec{R} = \int_{C_2} \vec{F} \cdot d\vec{R}}$



$\boxed{= \int_{C_3} \vec{F} \cdot d\vec{R}}$

- $\oint_C \vec{F} \cdot d\vec{R} = 0 = F(B) - f(A)$



Ex. Evaluate $\int_C \vec{F} \cdot d\vec{R}$

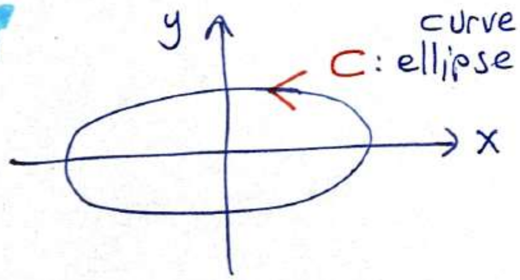
$$\vec{F}(x,y) = \left\langle (2x - x^2 y)e^{-xy} + \tan^{-1} y \,,\, \frac{x}{y^2+1} - x^3 e^{-xy} \right\rangle$$

a) on ellipse $9x^2 + 4y^2 = 36$

b) $\vec{R}(t) = \langle t^2 \cos(\pi t), e^{-t} \sin(\pi t) \rangle$, $0 \le t \le 1$.

$\vec{F}$ is conservative !

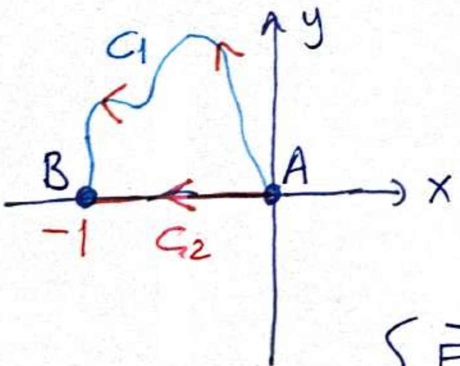
a)



$$\int_C \vec{F} \cdot d\vec{R} = \oint_C \vec{F} \cdot d\vec{R} = 0.$$

b) $A = \vec{R}(0) = \langle 0, 0 \rangle$

$B = \vec{R}(1) = \langle 1^2 \cos\pi, e^{-1} \sin\pi \rangle = \langle -1, 0 \rangle$



$y = 0$ $\quad 0 \le t \le 1$

$x = -t$

$\vec{R}_2(t) = \langle -t, 0 \rangle$

$\vec{R}_2'(t) = \langle -1, 0 \rangle$.

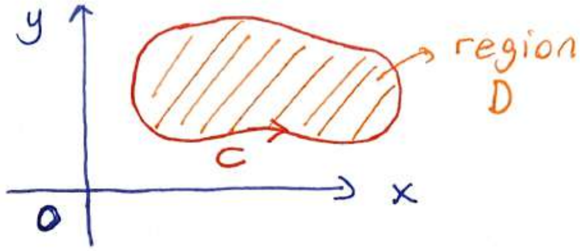
$$\int_{C_1} \vec{F} \cdot d\vec{R} = \int_{C_2} \vec{F} \cdot d\vec{R} = \int_{t_1}^{t_2} \vec{F}(R_2(t)) \cdot R_2'(t)\, dt$$

$$= \int_0^1 \langle 2(-t) - 0)e^0 + \tan^{-1}(0), \ldots \rangle \cdot \langle -1, 0 \rangle \, dt$$

$$= \int_0^1 2t \, dt = t^2 \Big|_0^1 = 1.$$

13.4 Green's Theorem

Green's theorem relates the line integral over a simple closed curve C to the double integral over a plane region D bounded by C.



The fundamental theorem of Calculus states

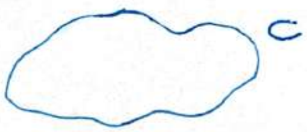
$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Similarly in Section 13.3,

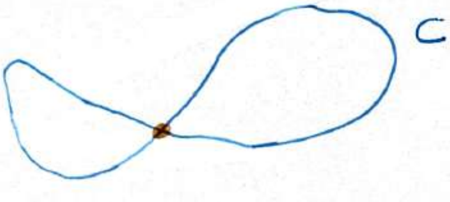
$$\int_C \nabla f \cdot d\vec{R} = f(B) - f(A)$$
$$= f(R(b)) - f(R(a)).$$

We will see an analogue to the fundamental theorem of calculus named Green's theorem.

Jordan curve : A closed, non-self-intersecting curve C.



a Jordan curve



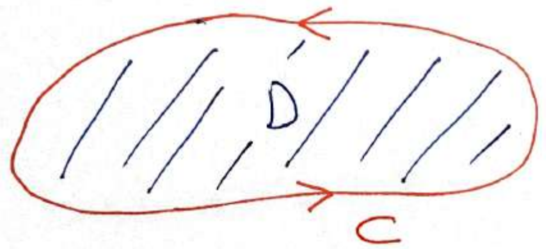
not a Jordan curve !

A self-intersecting curve C.

Green's Theorem (2D)

If $\vec{F} = \langle F_1, F_2 \rangle$ is a vector field defined on a simply connected region D bounded by a positively oriented Jordan curve C, Then

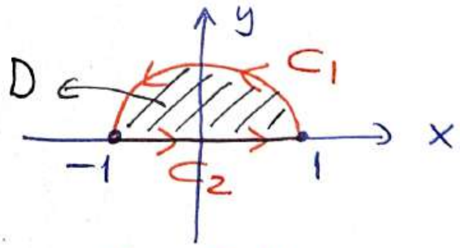
$$\boxed{\oint_C \vec{F} \cdot d\vec{R} = \iint_D \left(F_{2_x} - F_{1_y} \right) dA.}$$



Check Green's theorem is correct for

$$\int_C \vec{F} \cdot d\vec{R} = \int_C \underbrace{-y\,dx + x\,dy}^{\vec{F} \cdot d\vec{R}}$$

$F = \langle -y, x \rangle$
$\vec{dR} = \langle dx, dy \rangle$



$C = C_1 \cup C_2$

C_1: $\vec{R}_1(t) = \langle \cos t, \sin t \rangle \quad 0 \le t \le \pi$

$R_1'(t) = \langle -\sin t, \cos t \rangle$

C_2: $R_2(t) = \langle t, 0 \rangle \quad -1 \le t \le 1$

$R_2'(t) = \langle 1, 0 \rangle$

$$\int_C \vec{F} \cdot d\vec{R} = \int_{C_1 \cup C_2} \vec{F} \cdot d\vec{R} = \int_{C_1} \vec{F} \cdot d\vec{R} + \int_{C_2} \vec{F} \cdot d\vec{R}$$

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$$= \int_0^{\pi} \langle -\sin t, \cos t \rangle \cdot \langle \sin t, \cos t \rangle \, dt + \int_{-1}^{1} \langle 0, t \rangle \cdot \langle 1, 0 \rangle \, dt$$

$$= \int_0^{1} (\sin^2 t + \cos^2 t) \, dt + 0 = t \Big|_0^{\pi} = \pi .$$

Using Green's theorem:

$$\iint_D (F_{2x} - F_{1y}) \, dA = \iint_D [1 - (-1)] \, dA$$

$$= 2 \iint_D dA$$

$$= 2 \text{ Area}(D)$$

$$= 2 \cdot \frac{\pi \cdot 1^2}{2}$$

$$= \pi .$$

Remember:

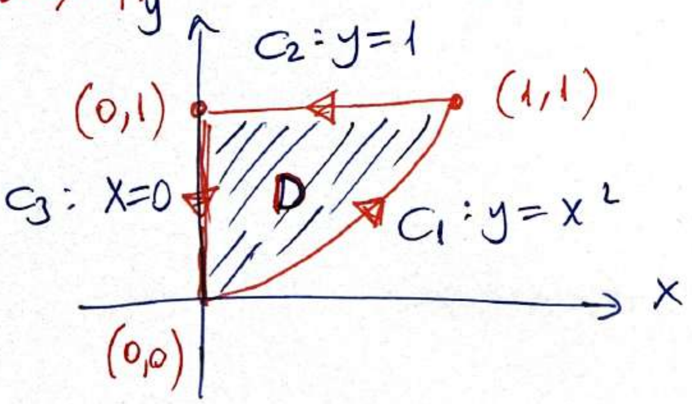
$$\int_C \vec{F}\cdot d\vec{R} =$$

- C is open:
  - $\vec{F}$ is not conservative: $= \boxed{\int_{t_1}^{t_2} \vec{F}(\vec{R}(t))\cdot \vec{R}'(t)\, dt}$ (13.2)
  - F is conservative: $= \boxed{F(q)-F(p)}$ (13.3) $\left\{ \vec{F} = \nabla f \right\}$
- C is closed: $= \boxed{\iint_D \left(F_{2_x} - F_{1_y}\right) dA}$ (13.4) Green's thm in 2D.

Ex: Find the work done on an object moving along C in the force field:

$$\vec{F}(x,y) = \langle \underbrace{x + xy^2}_{F_1},\ \underbrace{2(x^2y - y^2\sin y)}_{F_2} \rangle .$$

continuous, diff. on D.



$\underline{C = C_1 \cup C_2 \cup C_3}$

D: simply connected region with a positively oriented boundary

$$W = \oint_C \vec{F}\cdot d\vec{R}$$

$$= \iint_D \left(F_{2_x} - F_{1_y}\right) dA$$

$$= \iint_D \left(\frac{\partial}{\partial x}\left[2(x^2y - y^2\sin y)\right] - \frac{\partial}{\partial y}\left[x + xy^2\right]\right) dA$$

$$W = \iint_D (4xy - 2xy)\, dA$$

$$= \iint_D 2xy\, dA$$

$$= 2\int_{x=0}^{1}\int_{y=x^2}^{1} xy\, dy\, dx$$

$$= 2\int_0^1 x\frac{y^2}{2}\Big|_{y=x^2}^{y=1} dx$$

$$= 2\int_0^1 \frac{x}{2}\left((1)^2 - (x^2)^2\right) dx$$

$$= \frac{\cancel{2}}{\cancel{2}}\int_0^1 x\,(1 - x^4)\, dx$$

$$= \int_0^1 (x - x^5)\, dx$$

$$= \left(\frac{x^2}{2} - \frac{x^6}{6}\right)\Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{6}$$

$$= \frac{3-1}{6} = \frac{2}{6} = \frac{1}{3}.$$

Application.

We can use Green's theorem in both ways.

Consider the three vector fields

$$\frac{1}{2}\langle -y, x\rangle, \quad \langle -y, 0\rangle, \quad \langle 0, x\rangle.$$

Each of them holds

$$F_{2_x} - F_{1_y} = 1.$$

Then, for any simple connected domain D bounded by the positively oriented Jordan curve C, we get

$$\begin{aligned}
\text{Area}(D) &= \oint_C \frac{1}{2}\langle -y, x\rangle \cdot d\vec{R} \\
&= \oint_C \langle -y, 0\rangle \cdot d\vec{R} \\
&= \oint \langle 0, x\rangle \cdot d\vec{R} \\
&= \iint_D 1 \, dA
\end{aligned}$$

We can write

$$\text{Area}(D) = \frac{1}{2}\oint_C (x\,dy - y\,dx) = \iint_D dA.$$

Proof. Let $\vec{F}(x,y) = \langle -y, x \rangle$

continuously differentiable on D,

then, we can apply Green's theorem.

$$\Rightarrow \oint_C (-y\,dx + x\,dy) \qquad \left\{ \oint_C \underset{\langle -y,x\rangle}{\vec{F}} \cdot \underset{\langle dx, dy\rangle}{d\vec{R}} \right.$$

$$= \iint_D \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right) dA$$

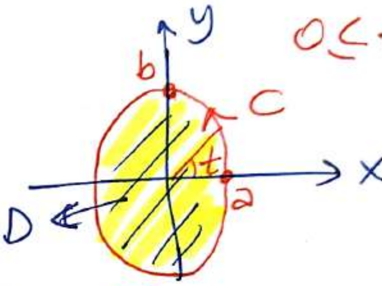
$$= \iint_D [1 - (-1)]\, dA$$

$$= 2\iint dA$$

$$= 2A$$

$$\Rightarrow \boxed{A = \frac{1}{2}\oint_C (-y\,dx + x\,dy).}$$

Ex. Find the area of the ellipse with semiaxis a and b centered at the origin.



$$\text{Area}(D) = \iint_D 1 \, dA$$

$$= \oint \frac{1}{2} \langle -y, x \rangle \cdot d\vec{R}$$

$$= \int_{t=t_1}^{t=t_2} \vec{F}(\vec{R}(t)) \cdot R'(t) \, dt$$

$$= \int_0^{2\pi} \frac{1}{2} \langle -b\sin t, a\cos t \rangle \cdot \langle -a\sin t, b\cos t \rangle dt$$

$$= \frac{1}{2} ab \int_0^{2\pi} (\underbrace{\sin^2 t + \cos^2 t}_{=1}) \, dt$$

$$= \frac{1}{2} ab \, (t) \Big|_0^{2\pi}$$

$$\text{Area}(D) = \pi ab.$$

Note: If C is negatively oriented, then

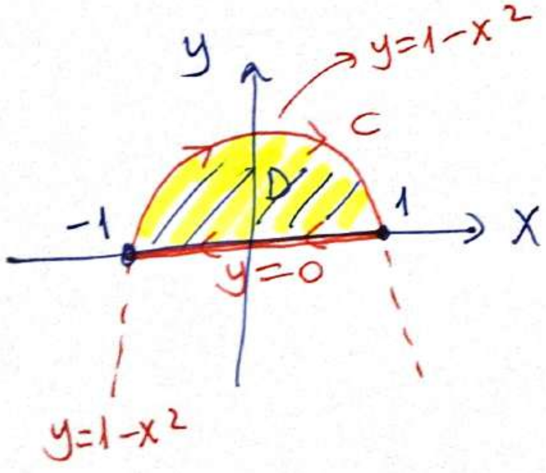
$$\boxed{\oint_C \vec{F} \cdot d\vec{R} = -\iint_D \left(F_{2_x} - F_{1_y}\right) dA}$$

$\vec{F} = \langle F_1, F_2 \rangle$

Ex. Find $\oint_C \vec{F} \cdot d\vec{R}$, where

$$\vec{F}(x,y) = \langle x^3 + xy^2 + y^2, -\cos y + x^2 y + x \rangle$$

and C is the closed piecewise curve defined by the positive arc of the parabola $y = 1 - x^2$ and the x-axis transversed clockwise.



C is negatively oriented

$$\boxed{\oint_C \vec{F} \cdot d\vec{R} = \iint_D \left(F_{1_y} - F_{2_x}\right) dA}$$

$$= -\iint_D \left(F_{2_x} - F_{1_y}\right) dA$$

$$\oint_C \vec{F} \cdot d\vec{R} = \iint_D \Big[\underbrace{(2xy+2y)}_{\substack{\| \\ F_{1_y}}} - \underbrace{(2xy+1)}_{\substack{\| \\ F_{2_x}}} \Big] dA$$

negatively oriented curve

$$= \iint_D (2y-1)\, dA$$

$$= \int_{x=-1}^{1} \int_{y=0}^{1-x^2} (2y-1)\, dy\, dx$$

$$= \int_{-1}^{1} (y^2 - y) \Big|_0^{1-x^2} dx$$

$$= \int_{-1}^{1} \Big[(1-x^2)^2 - (1-x^2) - 0 \Big] dx$$

$$= \int_{-1}^{1} \Big[(1 - 2x^2 + x^4) - 1 + x^2 \Big] dx$$

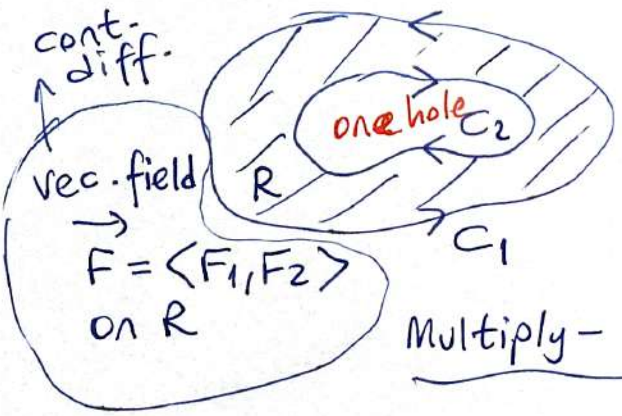
$$= \int_{-1}^{1} (-x^2 + x^4)\, dx$$

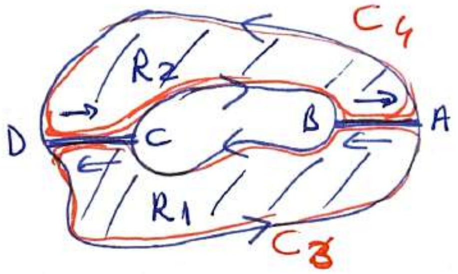
$$= \left(-\frac{x^3}{3} + \frac{x^5}{5} \right) \Big|_{-1}^{1} dx$$

$$= \left(-\frac{1}{3} + \frac{1}{5} \right) - \left(\frac{1}{3} - \frac{1}{5} \right)$$

$$= -\frac{2}{3} + \frac{2}{5} = \frac{-4}{15}$$

Green's theorem extends to regions with holes by dividing the region into simply connected subregions.





Multiply-connected region

one hole: doubly connected region

$$\iint_R \big(\underbrace{F_{2_x} - F_{1_y}}_{\text{briefly } \alpha} \big)\, dA = \iint_{R_1} \alpha \, dA + \iint_{R_2} \alpha \, dA$$

$$= \oint_{C_3} (F_1\,dx + F_2\,dy) + \oint_{C_4} (F_1\,dx + F_2\,dy)$$

The line integrals along A–B and C–D cancel with those along B–A and D–C, leaving only the boundary integrals along C_1 and C_2:

$$\iint_R (F_{2_x} - F_{1_y})\,dA = \oint_{C_1} (F_1\,dx + F_2\,dy) + \oint_{C_2} F_1\,dx + F_2\,dy.$$

Ex.

Show that the line integral

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = 2\pi \,,$$

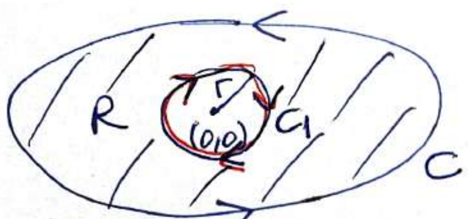
$= \beta$ (briefly)

where C is a simple closed curve enclosing the origin $(0,0)$.

$$F_1(x,y) = \frac{-y}{x^2+y^2} \,, \quad F_2(x,y) = \frac{x}{x^2+y^2}$$

Using Green's theorem:

$$\Rightarrow \iint_R (F_{2_x} - F_{1_y})\,dA = \oint_C (F_1\,dx + F_2\,dy) - \int_{C_1} (F_1\,dx + F_2\,dy)\,dA$$



a region R with one hole

$$F_{2_x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1.(x^2+y^2) - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$F_{1_y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(-1)(x^2+y^2) - (-y)\,2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \quad \underline{F_{2_x} - F_{1_y} = 0} \quad \text{for all } (x,y) \neq (0,0).$$

$$\Rightarrow \iint_R \overset{(F_{2x}-F_{1y})}{0} \cdot dA = 0 . \left(= \oint_C \frac{-y\,dx+x\,dy}{x^2+y^2} = -\oint_{C_1} \frac{-y\,dx+x\,dy}{x^2+y^2} \right)$$

Parametrization $-C_1$ (transversed counterclockwise):

$$\left.\begin{array}{ll} x = r\cos\theta & \Rightarrow dx = -r\sin\theta\,d\theta \\ y = r\sin\theta & \Rightarrow dy = r\cos\theta\,d\theta \end{array}\right\} \quad 0 \leq \theta \leq 2\pi$$

$$\Rightarrow \oint_{-C_1} \frac{-y\,dx+x\,dy}{x^2+y^2} = \int_0^{2\pi} \frac{-r\sin\theta(-r\sin\theta)+r\cos\theta(r\cos\theta)}{r^2(\sin^2\theta+\cos^2\theta)}\,d\theta$$

$$= \int_0^{2\pi} 1 \cdot d\theta = 2\pi .$$

$$\Rightarrow \oint_C \beta = \oint_{-C_1} \beta = 2\pi . \neq 0$$ (is not independent of path in R)

Alternate two forms for Green's theorem that generalize to $\mathbb{R}^3$:

(1) $\vec{F} = \langle F_1, F_2 \rangle$, $\vec{F} = \vec{F}(x,y)$,

$$\Rightarrow \text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & 0 \end{vmatrix} \qquad \left\{ \partial_x = \frac{\partial}{\partial x} \right\}$$

$$= \langle 0, 0, F_{2_x} - F_{1_y} \rangle$$

$$= 0\vec{i} + 0\vec{j} + (F_{2_x} - F_{1_y})\,\vec{k}$$

unit vector in the z-direction

$$\Rightarrow \iint_D (F_{2_x} - F_{1_y})\,dA = \iint_D (\text{curl}\,\vec{F} \cdot \vec{k})\,dA.$$

Also,

$$\oint_C \vec{F} \cdot d\vec{R} = \oint_C \vec{F} \cdot \frac{d\vec{R}}{ds}\,ds = \oint_C \vec{F} \cdot \vec{T}\,ds$$

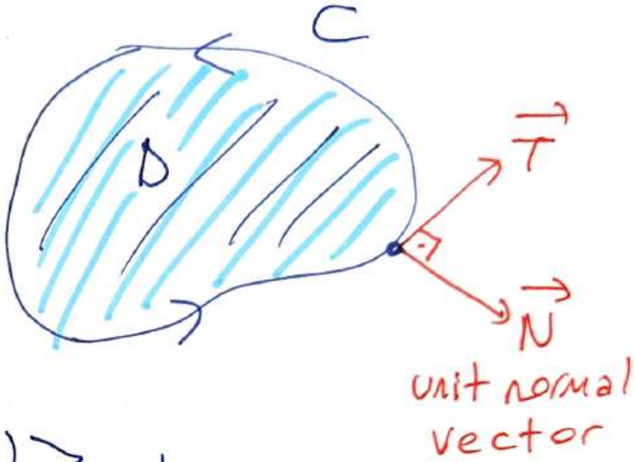
unit tangential vector to C

$$\oint_C \vec{F} \cdot d\vec{R} \overset{\text{(Green)}}{=} \iint_D (F_{2_x} - F_{1_y})\,dA$$

$$(\star)\ \left\{ \oint_C \vec{F} \cdot \vec{T}\,ds = \iint_D (\text{curl}\,\vec{F} \cdot \vec{k})\,dA \right.$$

This form $(\star)$ generalizes to Stoke's theorem for surfaces in $\mathbb{R}^3$. (see section 13.6)

2. $\vec{F} = \langle \vec{F_1}, \vec{F_2} \rangle$



s: arc length parameter

$\vec{R}(s) = \langle x(s), y(s) \rangle$

↳ position vector on C.

$\vec{T} = \vec{R}'(s) = \langle x'(s), y'(s) \rangle$,

$\vec{N} = \langle y'(s), -x'(s) \rangle$.

Green's :

$$\oint \vec{F} \cdot \vec{N}\, ds = \int_a^b \langle F_1, F_2 \rangle \cdot \langle y'(s), -x'(s) \rangle ds$$

$$= \int_a^b [F_1 \underbrace{y'(s)}_{\frac{dy}{ds}} - F_2 \underbrace{x'(s)}_{\frac{dx}{ds}}]\, ds$$

$$\Rightarrow \oint \vec{F} \cdot \vec{N}\, ds = \oint_C (-F_2\, dx + F_1\, dy)$$

$$= \iint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dx\, dy \quad \text{(Green)}$$

$$\Rightarrow \oint \vec{F} \cdot \vec{N}\, ds = \iint_D \operatorname{div} \vec{F}\, dA.$$

Green's theorem is applied to the normal derivative of a scalar function f, which is the directional derivative in the direction of the outward normal vector $\vec{N}$:

$$\frac{\partial f}{\partial n} = \nabla f \cdot \vec{N}.$$

Show that for a scalar function f with continuous partial derivatives in a simply connected region D bounded by curve C,

$$\iint_D \underbrace{\nabla^2 f}_{\substack{= \\ f_{xx}+f_{yy} \\ \text{(Laplacian of } f)}} \, dx\, dy = \oint_C \underbrace{\frac{\partial f}{\partial n}}_{\substack{= \\ \nabla f \cdot \vec{N} \\ \text{(normal derivative)}}} \, ds.$$

Let $u = -f_y$, $v = f_x$,

$$\Rightarrow \nabla^2 f = f_{xx} + f_{yy}$$

$$= v_x - u_y$$

$$\Rightarrow \iint_D \nabla^2 f \, dx\,dy = \iint_D (V_x - U_y)\, dx\,dy$$

$$= \oint_C (u\,dx + v\,dy) \quad (\text{Green})$$

$$= \oint_C [u\, x'(s) + v\, y'(s)]\, ds$$

$$= \oint_C [-f_y\, x'(s) + f_x\, y'(s)]\, ds$$

$$= \oint_C [f_x\, y'(s) - f_y\, x'(s)]\, ds$$

$$= \oint_C \underbrace{\langle f_x, f_y \rangle}_{\nabla f} \cdot \underbrace{\langle y'(s), -x'(s) \rangle}_{\vec{N}}\, ds$$

$$= \oint_C \nabla f \cdot \vec{N}\, ds$$

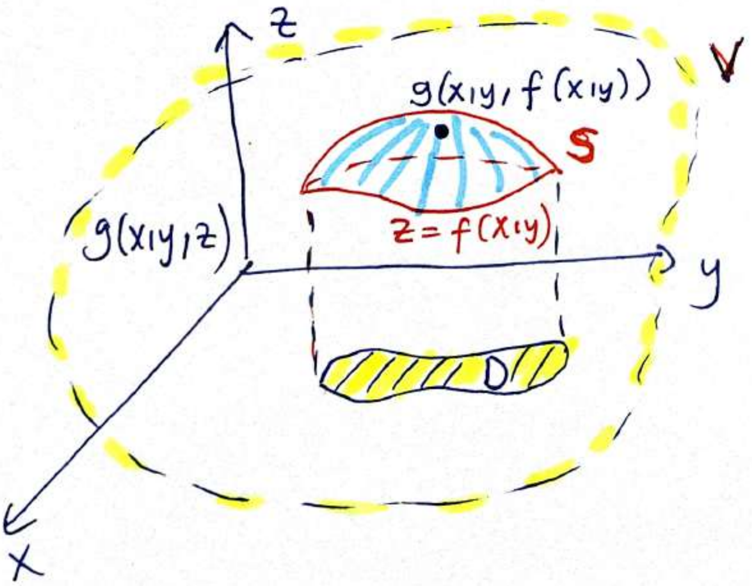
$$= \oint_C \frac{\partial f}{\partial n}\, ds$$

13.5 Surface Integrals

For a surface S defined by $z=f(x,y)$, with projection D on the xy-plane, and a continuous function $g(x,y,z)$ on $V \subset \mathbb{R}^3$, the surface integral of g over S is given by

$$\boxed{\iint_S \underbrace{g(x,y,z)}_{\text{on } V}\, dS = \iint_D \underbrace{g(x,y,f(x,y))\sqrt{(f_x)^2+(f_y)^2+1}}_{h(x,y)}\, dA}$$

$$\left\{ \begin{array}{l} \text{In 12.4 :} \\ \text{Surface Area} = \iint_S dS = \iint_D \sqrt{1+z_x^2+z_y^2}\, dA \end{array} \right\}$$



Ex. Evaluate $\iint_S z^2\, dS$, where S is the upper hemisphere $z=\sqrt{4-x^2-y^2}$.

$$g(x,y,z)=z^2 \quad , \quad f(x,y)=\sqrt{4-x^2-y^2} \, ,$$

$$\begin{aligned} dS &= \sqrt{1+(f_x)^2+(f_y)^2} \\ &= \sqrt{1+\left(-\frac{x}{z}\right)^2+\left(-\frac{y}{z}\right)^2} \\ &= \sqrt{1+\frac{x^2+y^2}{z^2}} \\ &= \sqrt{\frac{x^2+y^2+z^2}{z^2}} \\ &= \sqrt{\frac{4}{z^2}} = \frac{2}{z} \end{aligned}$$

$\underline{x^2+y^2+z^2=4}$:

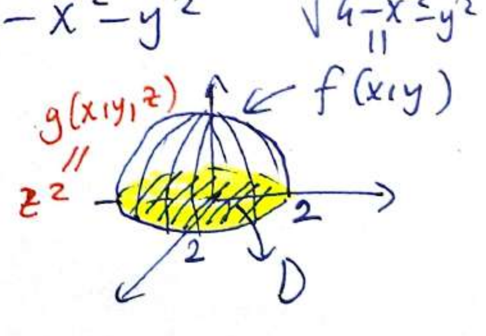
$$z_x=-\frac{F_x}{F_z}=-\frac{x}{z}$$

$$z_y=-\frac{F_y}{F_z}=-\frac{y}{z}$$

implicit differentials.

$$g(x,y,f(x,y)) = \left(\sqrt{4-x^2-y^2}\right)^2 = 4-x^2-y^2$$

$D: \quad x^2+y^2 \le 4$



$$\iint_S g(x,y,z)\, dA = \iint_D g(x,y,f(x,y))\sqrt{1+(f_x)^2+(f_y)^2}\, dA$$

$$= \iint_D (4-x^2-y^2)\,\frac{2}{\sqrt{4-x^2-y^2}}\,dA$$

$$= \iint_D 2\sqrt{4-x^2-y^2}\,dA$$

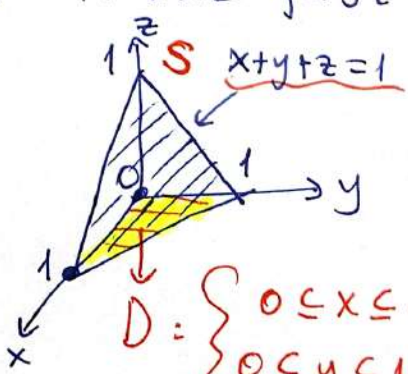
$$= -\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\sqrt{4-r^2}\,(-2r)\,dr\,d\theta$$

$$\left\{\begin{array}{l} u = 4-r^2 \\ du = -2r\,dr \\ -\int \sqrt{u}\,du \\ = -\frac{2}{3}u^{3/2} \end{array}\right\}$$

$$= -\frac{2}{3}\int_{\theta=0}^{2\pi}(4-r^2)^{3/2}\Big|_0^2\,d\theta$$

$$= -\frac{2}{3}\left(0-4^{3/2}\right)2\pi = \frac{32\pi}{3}.$$

Ex. Evaluate $\iint_S (2x+3y+z)\,dS$, where S is the portion of $x+y+z=1$ in the first octant.



$$g(x,y,z) = 2x+3y+z$$

$$D: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{cases}$$

$f(x,y) = 1-x-y$

$f_x = -1$

$f_y = -1$

$$dS = \sqrt{1+(f_x)^2+(f_y)^2} = \sqrt{1+(-1)^2+(-1)^2} = \sqrt{3}$$

$$g(x,y,f(x,y)) = 2x+3y+(1-x-y)$$
$$= x+2y+1$$

$$\Rightarrow \iint_S g(x,y,z)\,dS = \iint_D g(x,y,f(x,y))\sqrt{1+(f_x)^2+(f_y)^2}\,dA$$

$$= \iint_D (x+2y+1)\sqrt{3}\,dA$$

$$= \sqrt{3}\int_{x=0}^{1}\int_{y=0}^{1-x}(x+2y+1)\,dy\,dx$$

$$= \sqrt{3}\int_0^1 \left[(x+1)y + y^2\right]\Big|_{y=0}^{1-x} dx$$

$$= \sqrt{3}\int_0^1 \left[(1-x^2)+(1-x)^2\right] dx$$

$$= \sqrt{3}\int_0^1 (1-x^2+1-2x+x^2)\,dx$$

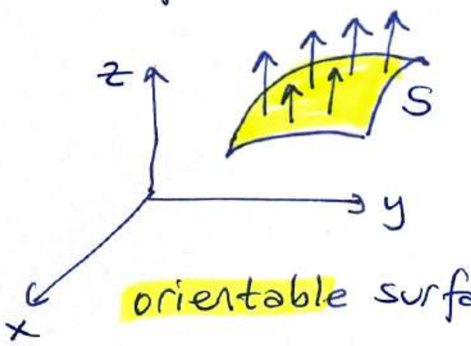
$$= \sqrt{3}\int_0^1 (2-2x)\,dx$$

$$= \sqrt{3}\,(2x-x^2)\Big|_0^1$$

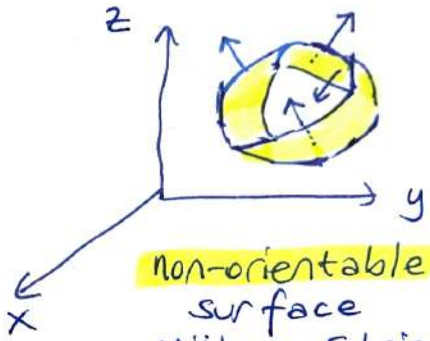
$$= \sqrt{3}\,(2-1)$$

$$= \sqrt{3}.$$

An orientable surface is a surface where is possible to make a consisten choice for the surface normal vector.



orientable surface



non-orientable surface
Möbius Strip

Let $F(x,y,z)$ be a vector field in $V \subseteq \mathbb{R}^3$.
Let S be an orientable surface inside V with unit normal $\vec{N}$. Then, the flux integral of $\vec{F}$ in S is given

$$\iint_S \vec{F} \cdot dS = \iint_S \underbrace{\vec{F} \cdot \vec{N}}_{\text{scalar function given } g(x,y,z)} ds$$

If S_1 is a surface with opposite orientation respect to S :

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot \vec{N_1}\, ds$$

$$= \iint_S \vec{F} \cdot (-\vec{N})\, ds = -\iint_S \vec{F} \cdot d\vec{s}$$

Let $\vec{F}(x,y,z)$ be a vector field in $V \subseteq \mathbb{R}^3$. Let S, defined by $z = f(x,y)$ for (x,y) in D, be a surface in V oriented upward. Then, the flux integral of $\vec{F}$ through S is determined by

Upward normal to $z = f(x,y)$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{N}\, dS$$

$$= \iint_D \vec{F}(x,y,f(x,y)) \cdot \overbrace{\frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1+(f_x)^2+(f_y)^2}}}^{\Rightarrow\ \vec{N}\ =} \overbrace{\sqrt{1+f_x^2+f_y^2}\, dA}^{dS\ =}$$

$$\Rightarrow \boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(x,y,f(x,y)) \cdot \langle -f_x, -f_y, 1\rangle\, dA}$$

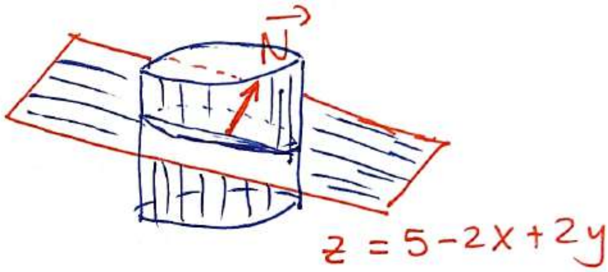
If S is oriented downward:

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(x,y,f(x,y)) \cdot \langle f_x, f_y, -1\rangle\, dA}$$

Ex.

Evaluate $\iint_S \langle 3x, 2y, z\rangle \cdot dS$,

where S is the portion of $2x-2y+z=5$ inside the cylinder $x^2+y^2=1$ oriented upward.



$f(x,y) = \underline{5-2x+2y}$

$$\left.\begin{array}{l} f_x=-2 \\ f_y=2 \end{array}\right\} \quad \langle -f_x, -f_y, 1\rangle = \langle 2,-2,1\rangle$$

$$\Rightarrow \vec{F}(x,y,f(x,y)) = \langle 3x, 2y, \underline{5-2x+2y}\rangle$$

$$\Rightarrow \iint_S \vec{F}\cdot dS = \iint_D \underbrace{\langle 3x, 2y, z\rangle}_{\vec{F}(x,y,f(x,y))} \cdot \underbrace{\langle 2,-2,1\rangle dA}_{dS}$$

$$= \iint_D (6x-4y+5-2x+2y)\, dA$$

$$= \iint_D (4x-2y+5)\, dA$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (4r\cos\theta - 2r\sin\theta)\, r\, dr\, d\theta + 5\iint_D 1\, dA$$

$$= \int_0^{2\pi} (4\cos\theta - 2\sin\theta)\frac{r^3}{3}\Big|_0^1 d\theta + 5\iint_D 1\, dA$$

$$= \frac{1}{3}\left(4\sin\theta + 2\cos\theta\right)\Big|_{0}^{2\pi} + 5\iiint_D 1.dA$$

$$= \underbrace{4(0-0)+2(1-1)}_{=0} + 5\iiint_D dA$$

$$= 5\iiint_D dA$$

$$= 5\ \text{Area}(D)$$

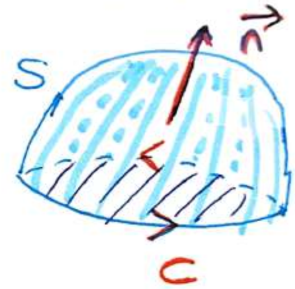
$$= 5\,(\text{Area of circle with radius one})$$
$$= \pi\cdot(1)^2$$

$$\Rightarrow \iint_S \vec{F}\cdot dS = 5\pi.$$

13.6 Stokes Theorem

Let S be an oriented surface whose boundary is the curve C oriented according to $\vec{n}$.

When the thumb points in the direction of $\vec{n}$, the fingers curl in the forward direction around C.



Let $\vec{F} = \langle f_1, f_2, f_3 \rangle$ be a vector field with domain V. Let S be in V. Then,

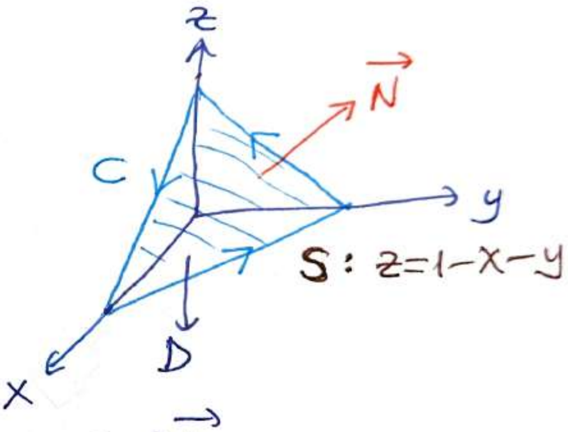
$$\boxed{\oint_C \vec{F} \cdot d\vec{R} = \iint_S \nabla \times \vec{F} \cdot d\vec{S}}$$

Green's theorem is Stoke's theorem in 2D, where S is D (surface on xy-plane) $\vec{n}$ is $\vec{k} = \langle 0, 0, 1 \rangle$, C is oriented with $\vec{k}$ (positively oriented Jordan curve) and $\vec{F} = \langle F_1, F_2, 0 \rangle$.

$$\int_C \vec{F} \cdot d\vec{R} = \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_D \langle 0, 0, F_{2_x} - F_{1_y} \rangle \cdot \langle 0, 0, 1 \rangle \, dA$$

$$= \iint_D \left(F_{2_x} - F_{1_y} \right) dA$$

Ex. Let S be the portion of $x+y+z=1$ in the first octant and let C be the boundary of S transversed counterclock-wise when viewed from above. Let $\vec{F} = \langle -\frac{3}{2}y^2, -2xy, yz \rangle$. Use Stoke's theorem to find $\oint_C \vec{F} \cdot d\vec{R}$.



$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -\frac{3}{2}y^2 & -2xy & yz \end{vmatrix}$$

$$= (z-0)\vec{i} - (0-0)\vec{j} + (-2y+3y)\vec{k}$$

$$= \langle z, 0, y \rangle.$$

$$\oint_C \vec{F} \cdot d\vec{R} = \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_S \langle z, 0, y \rangle ds$$

According to flux integral in 13.5 :

$f(x,y) = 1-x-y$, $\vec{N} = \langle -f_x, -f_y, 1 \rangle = \langle 1, 1, 1 \rangle$,

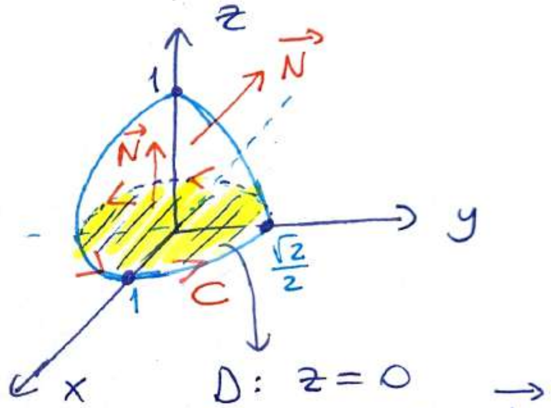
$$\iint_S \langle z, 0, y \rangle \cdot d\vec{S} = \iint_D \langle 1-x-y, 0, y \rangle \cdot \langle 1, 1, 1 \rangle dA$$

$$= \iint_D (1-x-y+y)\, dA = \iint_D (1-x) dA$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (1-x) dy\, dx = \int_0^1 (1-x)\, y \Big|_0^{1-x} dx$$

$$= \int_0^1 (1-x)^2 dx = \int_0^1 (1-2x+x^2) dx = \frac{1}{3}.$$

Ex. Evaluate $\iint_S \nabla\times\vec{F}\cdot d\vec{S}$ where $\vec{F}=\langle x, y^2, ze^{xy}\rangle$ and S is the portion of the paraboloid $z=1-x^2-2y^2$ with $z\geq 0$ oriented upward.



D: $z=0$

$\vec{N}=\langle 0,0,1\rangle=\vec{k}$

$$\iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{R}$$

$$= \iint_D \nabla\times\vec{F}\cdot \underbrace{d\vec{A}}_{\vec{k}\,dA}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x & y^2 & ze^{xy} \end{vmatrix} = (xze^{xy}-0)\vec{i} - (yze^{xy}-0)\vec{j} + (0-0)\vec{k}$$

$$\nabla\times\vec{F} = \langle xze^{xy}, -yze^{xy}, 0\rangle.$$

$$\iint_D \langle xze^{xy}, -yze^{xy}, 0\rangle\cdot d\vec{A}$$

$$= \iint_D \langle 0,0,0\rangle\cdot\langle 0,0,1\rangle\, dA$$

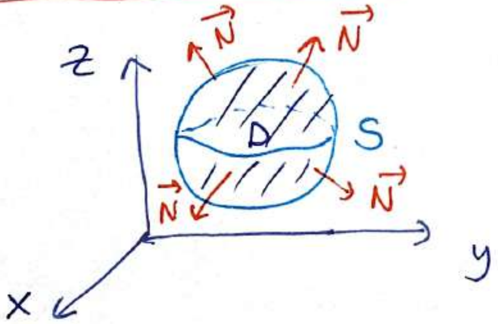
"$z=0$"
xy-plane

$$=0$$

13.7. Divergence Theorem

Let $\vec{F} = \langle F_1, F_2, F_3 \rangle$ a vector field in $V \subseteq \mathbb{R}^3$.
Let S a closed surface in V, oriented outward.
Let D be the interior of S. Then,

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div} \vec{F}\, dV = \iiint_D (\nabla \cdot \vec{F})\, dV}$$



Ex. Evaluate $\iint_S \langle xy^2, yz^2, zx^2 \rangle \cdot d\vec{S}$, where
S is the sphere $x^2+y^2+z^2=4$ oriented outward.

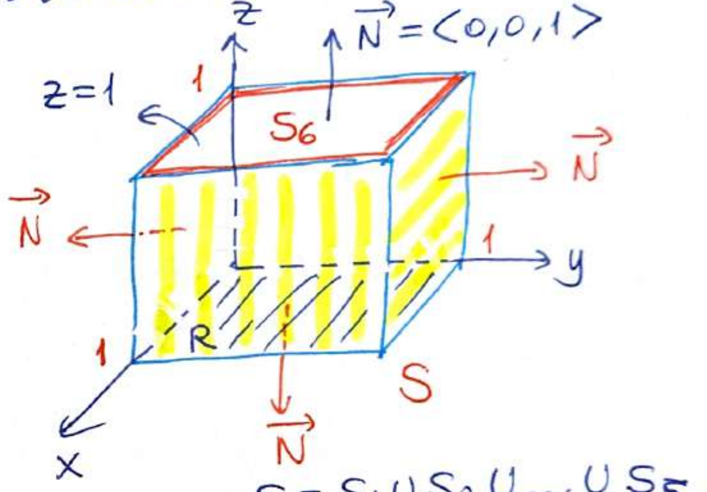
$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div} \vec{F}\, dV = \iiint_D (x^2+y^2+z^2)\, dV$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{2} \rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \frac{\rho^5}{5} \Big|_0^2 \sin\phi \, d\phi \, d\theta$$

$$= \frac{128}{5}\pi.$$

Ex. Evaluate $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F}=\langle x^2,y^2,z^2\rangle$ and S is the open box (no top) oriented toward the outside.



$S=S_1\cup S_2\cup\ldots\cup S_5$

The surface is open.
In order to use divergence theorem we have to add and substract the integral on the top face S_6 oriented upward.

$$\underset{\substack{S\\ \parallel \\ S_1\cup S_2\cup\ldots\cup S_5}}{\iint} \vec{F}\cdot d\vec{S} = \iint_{S+S_6} \vec{F}\cdot d\vec{S} - \iint_{S_6} \vec{F}\cdot d\vec{S}$$

$$= \iiint_D \operatorname{div}\vec{F}\, dV - \iiint_{S_6} \operatorname{div}\vec{F}\, dV .$$

Instead of solving 5 flux integrals, we solve a flux integral and a triple integral.

$$\iiint_D \operatorname{div}\vec{F}\, dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (2x+2y+2z)\, dz\, dy\, dx$$

$$\overset{?}{=} 6\int_0^1\int_0^1\int_0^1 x\, dx\, dy\, dz$$

$$= 6\int_0^1\int_0^1 \frac{x^2}{2}\Big|_0^1 dy\, dx = 6\cdot\frac{1}{2} = 3.$$

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$$\iint_{S_6} \vec{F} \cdot d\vec{S} = \iint_R \langle x^2, y^2, \underset{z=1}{\underset{\downarrow}{1}}{}^2 \rangle \cdot \langle 0,0,1 \rangle \, dA$$

$$\left\{ \iint \vec{F}(x,y,f(x,y)) \cdot \langle -f_x, -f_y, 1 \rangle \, dA \quad z = f(x,y) = 1 \right\}$$

$$= \iint_R 1 \cdot dA = 1$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div} \vec{F} \, dV - \iiint_{S_6} \operatorname{div} F \, dV$$

$$= \iiint_D \operatorname{div} F \, dV - \iint_{S_6} \vec{F} \cdot d\vec{S}$$

$$= 3 - 1$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = 2$$

EX. Find the normal vector of the following surface

$S: \quad \vec{r}(x,y) = \langle x, y, f(x,y) \rangle .$

$$\vec{r}_x = \langle 1, 0, f_x \rangle ,$$

$$\vec{r}_y = \langle 0, 1, f_y \rangle ,$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}$$

$$\vec{r}_x \times \vec{r}_y = (0 - f_x)\vec{i} - (f_y - 0)\vec{j} + (1-0)\vec{k}$$

$$\vec{r}_x \times \vec{r}_y = \langle -f_x , -f_y , 1 \rangle$$

$$\| \vec{r}_x \times \vec{r}_y \| = \sqrt{(-f_x)^2 + (-f_y)^2 + (1)^2}$$

$$= \sqrt{(f_x)^2 + (f_y)^2 + 1}$$

$$\Rightarrow \boxed{\underset{\substack{\downarrow \\ \text{normal} \\ \text{vector} \\ \text{of } S}}{\vec{N}} = \frac{\vec{r}_x \times \vec{r}_y}{\| \vec{r}_x \times \vec{r}_y \|} = \frac{\langle -f_x, -f_y, 1 \rangle}{\sqrt{(f_x)^2 + (f_y)^2 + 1}}}$$

Green's Theorem:

$$\oint_C \underset{\langle F_1, F_2\rangle}{\underset{\parallel}{\vec{F}}} \cdot d\underset{\langle dx, dy\rangle}{\underset{\parallel}{\vec{R}}} = \iint_D (F_{2_x} - F_{1_y})\, dA$$

For $\oint_C \underset{\langle -F_2, F_1\rangle}{\underset{\parallel}{\vec{F}}} \cdot \underset{\langle dx, dy\rangle}{\underset{\parallel}{\vec{N}}}\, ds \overset{?}{=} \iint_D (F_{1_x} + F_{2_y})\, dA$

using Green's Theorem.

Since $\langle F_1, F_2\rangle$ replaces with $\langle -F_2, F_1\rangle$,

$(F_{2_x} - F_{1_y})$ replaces with $(F_{1_x} + F_{2_y})$. !